Discrete mathematics and algorithms

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Main topics

- Discrete optimization
- Limits of combinatorial structures
- Random discrete structures
- Classification by multifractal analysis techniques

Discrete optimization

- Frédéric Meunier: Optimization techniques for bike-sharing systems
- Éric Colin de Verdière, Frédéric Meunier: Embedded T-joins

Bike-sharing systems

Abound in optimization problems.

- ⋆ Station location
- ⋆ Fleet dimensioning
- ⋆ Inventory setting
- ⋆ Rebalancing incentives
- ⋆ Bike repositioning

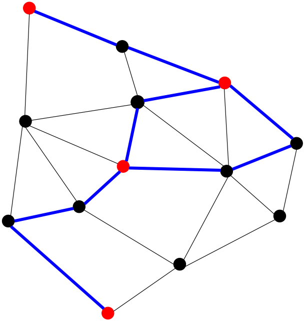
Except the first problem, they all have static/dynamic versions.

Contribution:

- Efficient algorithm for the 1-truck static repositioning problem
- Survey

Project: Study of the 1-truck dynamic repositioning problem

Embedded T-joins



Graph $G = (V, E)$, subset $T \subseteq V$

T-join = subgraph whose odd degree vertices are the elements in T

Useful for various combinatorial optimization problems, e.g., postman tour, max-cut.

Can we compute efficiently optimal T-joins of an embedded graph, with a homological constraint?

Limits of combinatorial structures

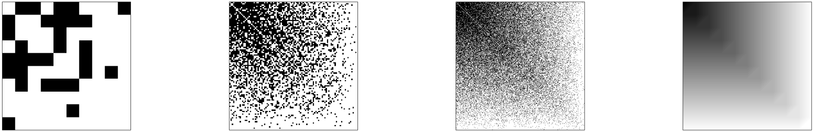
- Jean-François Delmas: Statistical approximation of large networks by graphons.
- Xavier Goaoc, Alfredo Hubard: Combinatorial limits of order types of finite point sets

Graphons and large networks

Graphon W: symmetric measurable function $[0,1]^2 \to [0,1]$

Random graph $G_n(W)$ sampled from W:

- vertex set: $[n]$
- ij: edge of $G_n(W)$ with probability $W(X_i, X_j)$, where $(X_i : i \in \mathbb{N}^*)$ are uniformly iid on $[0,1]$.



Convergence and fluctuations

Normalized degree function of W: $D(x) = \int_0^1 W(x,y)\mathrm{d}y$

Empirical cumulative distribution function of $G_n(W)$:
$\Pi_n(y) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{D_i^{(n)} \leqslant y\right\}}$
where $D_i^{(n)}$ is the normalized degree of vertex i in $G_n(W)$.

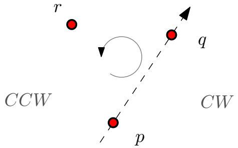
Theorem (Delmas, Dhersin, Sciauveau 2018)

If D is increasing, we have

- $\sup_y |\Pi_n(y) - D^{-1}(y)| \xrightarrow[n\to+\infty]{a.s.} 0$ *(almost sure convergence)*
- $\left(\sqrt{n}\left(\Pi_n(y) - D^{-1}(y)\right)\right)_y \xrightarrow[n\to+\infty]{(fdd)} \chi$
  (convergence of finite-dimensional distributions)
  where $\chi = (\chi_y)_y$ is a centered Gaussian process.

From geometry to combinatorics and back

Combinatorial limits (Razborov + Lovász et al.) of order types of finite point sets



Q: Can measures in the plane be to limits of order types what graphons are to dense graphs?

A: NO in general, yet, we observe some rigidity.

Results

Theorem (Goaoc, Hubard, de Joannis de Verclos, Sereni, Volec, 2018+)

Two absolutely continuous measures in the plane induce the same limit of order types if and only if one is the push-forward of the other one by a projective transformation.

Provide first non trivial results on the cases $k = 5, 6, +\infty$ of (Sylvester, Erdős):

$$\inf_{\mu} \mathbb{P}_{\mu}(k \text{: points form a convex polygon})$$

Random discrete structures

Matthieu Fradelizi, Xavier Goaoc, Alfredo Hubard: Random polytopes

Bárány-Larman theorem

Given a measure μ on $\mathbb{R}^d$, we define

- $\star$ a random polytope

$$P_n^\mu = \mathrm{conv}(x_1, \ldots, x_n),$$

where (x_i) are uniformly independently drawn from μ;

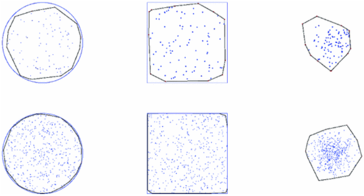
- $\star$ the wet parts

$$W_t^\mu = \{x \in \mathbb{R}^d : \exists \text{ a halfspace } h \ni x \text{ s.t. } \mu(h) \leqslant t\}.$$

Theorem (Bárány-Larman 1988)

If μ is uniform on a convex body, then

$$\frac{n}{e} \mathrm{Vol}(W_{1/n}^\mu) \leqslant \mathbb{E}(|V(P_n^\mu)|) \leqslant 2n \mathrm{Vol}(W_{c/n}^\mu).$$



Random polytopes and wet parts

Ongoing project: to what extent can this be generalized to any measure?

Theorem (Bárány, Fradelizi, Goaoc, Hubard, Rote 2018+)

For any measure μ, we have

$$\frac{n}{e}\,\mathrm{Vol}(W^{\mu}_{1/n}) \leqslant \mathbb{E}(|V(P^{\mu}_{n})|) \leqslant 2n\,\mathrm{Vol}(W^{\mu}_{c\log n/n}),$$

and both inequalities are sharp.

(Proof uses ε-nets.)

There are still many questions left:

- does the B-L bound hold for log-concave measures?
- which behavior is more typical among integers?

Multifractal analysis

Stéphane Jaffard: Classification by multifractal analysis techniques

Multifractal spectrum

Let $f \in L^{\infty}_{loc}(\mathbb{R}^d)$.

$f \in C^{\alpha}(x_0)$ if there exists a polynomial P such that, for r small enough, $\sup_{B(x_0,r)} |f(x) - P(x - x_0)| \leqslant Cr^{\alpha}$.

$h_f(x_0) = \sup\{\alpha\colon f \in C^{\alpha}(x_0)\}$.

Multifractal spectrum of f:

$$D_f(H) = \dim_B\{x\colon h_f(x) = H\},$$

where $\dim_B$ is the fractal dimension.

Multifractal analysis of paintings: Van Gogh challenge